THE THEOREM OF THREE MOMENTS.

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THE theorem expressing the relationship between the moments of flexure of a straight elastic girder at three successive points of support, was first published by Clapeyron in the Comptes Rendus, 1857. The investigation contemplated a girder of uniform cross section, whose points of support were situated upon the same level at any unequal distances apart, and loaded with any uniform loading between the successive points of support, but that loading of different intensity in the different spans into which the points of support divide the girder. The formula has since been generalized by the labors of Winkler, Bresse, Heppel, Weyrauch and others so as to include all cases; viz: when the loading is distributed in any manner whatever, either continuously or discontinuously; when the cross section of the girder changes uniformly or at intervals; when the points of support are at any different heights, and when the girder is fixed at the end supports in such a manner that it can or cannot change its direction at one or both of those points. The formula supposes that the elastic limits are in no case surpassed, either by too great loading or too great difference in height between the points of support, or both.

It is proposed in the present paper to obtain, in a direct manner, a new and simplified equation to express this relationship in its most general form, and to point out at the same time what may be regarded as the proper significance of the quantities appearing in the equation.

Let us assume the ordinary fundamental equations applicable to elastic girders under the action of vertical forces as already proven: they may be stated as follows:

Let O be any point of the girder at which we are to express either the shearing stress, the bending moment, the curvature, the slope, or the deflection.

Let $z =$ the distance from O as an origin to the point of application of any vertical force P.

Let $x =$ the distance from O as an origin to the point of application of any bending moment M.

Let $r =$ the radius of curvature of the deflected girder at O.

Let $I =$ the moment of inertia of the cross section of the girder about its neutral axis at any point.

Let $E =$ the modulus of elasticity of the material, supposed to be constant.

Then the relations we propose to assume are:

$$\left.\begin{array}{l}\text{Force applied at any point is } P. \\ \text{Shearing stress at } O \text{ is } S = \Sigma\,(P). \\ \text{Bending moment at } O \text{ is } M = \Sigma\,(Pz).\end{array}\right\} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

$$\left.\begin{array}{l}\text{Curvature at } O \text{ is } 1 \div r = R = M \div EI. \\ \text{Slope at } O \text{ is } T = \Sigma\,(R) = \Sigma\,(M \div EI). \\ \text{Deflection at } O \text{ is } D = \Sigma\,(Rx) = \Sigma\,(Mx \div EI).\end{array}\right\} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (2)$$

It is evident that P bears the same kind of relation to M in (1), that $M \div EI$ does to D in (2). In order to comprehend what the relationship is, it is only necessary to conceive that vertical ordinates be laid off at each point of the span equal to M; the locus of their extremities has been called the "curve of the actual moments," and it is known that it is the same as the catenary of the same depth which supports the weights P. Now from this curve, another curve (or polygon as the case may be) can be described, whose ordinates are equal to $R = M \div EI$, which may be called the "curve of the effective moments," since the amount of bending (*i. e.* difference of slope) and the deflection are dependent upon R. If this curve of effective moments be regarded as the surface of some species of homogenous loading whose depth is R, and a third curve be drawn such that its ordinates are the moments which would be produced by such loading, then the third curve is the "curve of actual deflections," and is the shape which the deflected girder will assume under the action of the loads P. By well-known graphical methods, these curves can all be constructed without the use of algebraic processes, with at least sufficient accuracy for practical computations.

It should be noticed that M is used in two senses: it may signify the ordinate of the moment curve first mentioned; or, it may signify a part of the "moment area" included between that curve and the span and having an horizontal width of one unit. It is in this last sense that it is used in (2), so that if it be desirable to estimate the dimension of the terms in any equation, regard must be had to this fact. It will readily appear in which sense M is used in the expressions hereafter employed, even though both are found in the same equation.

Let $I_0 =$ the moment of inertia about its neutral axis of some particular cross section, which is assumed as the standard of comparison.

Let $i = I_0 \div I =$ the ratio of the standard moment of inertia to that at any other cross section.

Let a, c, a' be three successive piers of a continuous girder of several unequal spans, in the order of their position, c being the intermediate pier.

Let magnitudes in the right hand span be distinguished from the corresponding magnitudes in the left, by primes.

Let l and l' be the length of the spans.

Let t and t' be the trigonometrical tangents of the acute angles at c between the horizontal and the line tangent to the deflection curve.

Let y be the ordinate of any point of the girder above some datum level, *i. e.* above the axis of x.

Then, in case of deflections so small as those occurring in elastic girders, we have sensibly

$$D_a = y_a - y_c - lt \quad , \quad D'_a = y'_a - y_c - l't' , \quad . \quad . \quad . \quad . \quad (3)$$

in which the deflections are reckoned from the tangent at c to the deflected girder. Also, the equation of deflections (2) may be written,

$$DEI_0 = \Sigma \left(Mix \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which any ordinate M of the moment curve in the span l, may be regarded as consisting, in the aggregate, of three parts: viz, M_1 dependent upon the moment M_a at the pier a, M_2 caused by the weights P in the span itself, and M_3 dependent upon the moment M_c at c. M_1 and M_3 are the effect at any point in the span of the weights P, which are applied to spans other than l. We therefore obtain by (3) for the two spans under consideration, when the summation is extended from c, the intermediate pier, to each of the piers a and a'.

$$\left. \begin{aligned} EI_0 \left(y_a - y_c - lt \right) &= \Sigma_c^a \left[\left(M_1 + M_2 + M_3 \right) ix \right] \\ EI_0 \left(y'_a - y_c - l't' \right) &= \Sigma_c^{a'} \left[\left(M'_1 + M'_2 + M'_3 \right) i'x' \right] \end{aligned} \right\} \quad . \quad . \quad . \quad (5)$$

in which x and x' are measured respectively from a and a' towards c.

Now the effect of M_a, which is a bending moment at a due to loads at the left of a, is a moment M_1 which uniformly decreases from a to c, as may be readily seen when we have regard to the fact that M_a is a couple which is held in equilibrium by its increasing (or decreasing) the vertical reaction at c.

The moment due to a vertical reaction at c increases uniformly from c to a, hence

$$M_1 = M_a \left(l - x \right) \div l, \text{ and } M_3 = M_c x \div l, \quad . \quad . \quad . \quad . \quad (6)$$

8

Let $$\bar{x} = \Sigma_c^a (Mix) \div \Sigma_c^a (Mi) , \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

then is $\bar{x}$ the distance from a to the centre of gravity of the effective moment area due to the moments M; $\bar{x}$ can be found with ease by a graphical process.. Then, from (6) and (7),

$$\left. \begin{array}{l} \bar{x}_1 = \int_c^a i\,(l-x)\,xdx \div \int_c^a i\,(l-x)\,dx , \\[2mm] \bar{x}_3 = \int_c^a i\,x^2 dx \div \int_c^a i\,xdx . \end{array} \right\} \quad . \quad . \quad . \quad . \quad (8)$$

Again, let $$i_1 = \Sigma_c^a (M_1 i) \div \Sigma_c^a (M_1) , \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

then is i_1 an average value of i for the moment area due to M_a. Then from (6) and (9),

$$\left. \begin{array}{l} i_1 = \int_c^a i\,(l-x)\,dx \div \int_c^a (l-x)\,dx , \\[2mm] i_3 = \int_c^a i\,x\,dx \div \int_c^a xdx. \end{array} \right\} \quad . \quad . \quad . \quad . \quad (10)$$

In similar manner it may be convenient to let $\bar{x}_2$ be the distance of the center of gravity of the effective moment area due to the weights P within the span, and to let i_2 denote the average value of i for the same area; but it is not possible in general to write integrals expressing the values of $\bar{x}_2$ and i_2 in terms of x, until the distribution of the loading is given: they are to be found in any case from equations of the form of (7) and (9).

When the value of i varies discontinuously, it is necessary to divide the limits of the integrals in (8) and (10), so that instead of a single integral we have the sum of several, each extending over a single portion of the span in which i varies continuously.

Once more, $$\left. \begin{array}{l} \Sigma_c^a (M_1) = \tfrac{1}{2} M_a l \\[1mm] \Sigma_c^a (M_3) = \tfrac{1}{2} M_c l, \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

as appears from previous statements, for these are the expressions for the moment areas due to the moments M_a and M_c respectively.

Equations (5) may now be written

$$\left. \begin{array}{l} EI_0\,(y_a - y_c - lt) = i_2\,\bar{x}_2\,\Sigma_c^a (M_2) + \tfrac{1}{2}l\,(M_a i_1 \bar{x}_1 + M_c i_3 \bar{x}_3) \\[1mm] EI_0\,(y'_a - y'_c - lt') = i'_2\,\bar{x}'_2\,\Sigma_c^{a'} (M'_2) + \tfrac{1}{2}l'\,(M'_a i'_1 \bar{x}'_1 + M'_c i'_3 \bar{x}'_3) \end{array} \right\} \cdot \quad (12)$$

Let $$t + t' = T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

then is T a known constant, for if the girder is straight at c before deflection, then $T = 0$, and in any case T is the acute angle between the spans l and l' when that is small enough to be regarded as sensibly equal to its tangent.

Now divide equations (12) by l and l' respectively and add, then by (13)

$$EI_0 \left[\frac{y_a - y_c}{l} + \frac{y'_a - y'_c}{l'} + T \right] = \frac{i_2\,\bar{x}_2}{l} \Sigma_c^a (M_2) + \frac{i'_2\,\bar{x}'_2}{l'} \Sigma_c^{a'} (M'_2)$$

$$+ \tfrac{1}{2} \left[M_a i_1 \bar{x}_1 + M_c i_3 \bar{x}_3 + M'_c i'_3 \bar{x}'_3 + M'_a i'_1 \bar{x}'_1 \right] \quad . \quad . \quad . \quad . \quad (14)$$

This equation expresses *the theorem of three moments* in its most general form : the only unknown quantities in it, for a given girder and given loading, are the moments at the piers $a\ c\ a'$. When there is no constraint at c, then $M_c = M'_c$; and in any case, $M_c - M'_c = C$ the couple introduced by the constraint at c.

The complexity of this formula, as obtained heretofore, has been due to the fact that the first two terms of the second member have been expressed in terms of the weights P, and no adequate method has been proposed for expressing and interpreting the remaining quantities in the second member, with the exception of the moments.

Let us now derive from (14) the equation expressing the theorem in case of an unconstrained girder having a uniform cross section, when the two first terms are stated in terms of the weights P.

In this case $i = 1$, and we have to determine $\bar{x}_2 \Sigma_c^a (M_2)$. Suppose that the area $\Sigma_c^a (M_2)$ is the sum of parts due to several weights P, then the part due to a single weight is

$$M = Pz (l - z) \div l , \quad \ldots \ldots \ldots \quad (15)$$

and this may be taken as the height of a triangular moment area whose base is l due to the weight P. This triangle whose area $= \frac{1}{2} M_2 l$ is the part of the moment area due to P, and in computing $\bar{x}_2 \Sigma_c^a (M_2)$ we must find the product of its area by the distance x of its center of gravity from a. Now $x = \frac{1}{3} (l + z)$

$$\therefore \quad \bar{x}_2 \Sigma_c^a (M_2) = \frac{1}{6} \Sigma_c^a [P (l^2 - z^2) z] \quad \ldots \ldots \ldots \quad (16)$$

Also in this case, $\quad \bar{x}_1 = \frac{1}{3} l , \quad \bar{x}_3 = \frac{2}{3} l ,$

$$\therefore \quad 6 EI \left[\frac{y_a - y_c}{l} + \frac{y_a - y_c}{l'} + T \right] = \frac{1}{l} \Sigma_c^a \left[P (l^2 - z^2) z \right] - \frac{1}{l'} \Sigma_c^{a'} \left[P (l'^2 - z'^2) z' \right]$$

$$+ M_a l + 2 M_c (l + l') + M_a l' , \quad \ldots \ldots \ldots \quad (17)$$

a well known form.

Again, if the girder is straight at c, the piers on the same level, and the cross section constant, we have

$$6 \left[\frac{1}{l} A_2 \bar{x}_2 + \frac{1}{l'} A'_2 \bar{x}'_2 \right] = M_a l + 2 M_c (l + l') + M_a l' ,$$

a form of the equation first given by Professor Chas. E. Greene, 1875, in which A_2 and A'_2 are the moment areas due to the applied weights.